Year: 1986

Mathematical instanton bundles on P2n+1

Okonek, C ; Spindler, H

DOI: https://doi.org/10.1515/crll.1986.364.35

Originally published at:
Okonek, C; Spindler, H (1986). Mathematical instanton bundles on P2n+1. Journal für die Reine und Angewandte Mathematik, 364:35-50.
DOI: https://doi.org/10.1515/crll.1986.364.35

Mathematical instanton bundles on $\mathbb{P}^{2n+1}$

By *Christian Okonek* and *Heinz Spindler**) at Göttingen

0. Introduction

A mathematical instanton bundle with quantum number k on $\mathbb{P}^3$ is by definition a holomorphic rank-2 bundle E with Chern polynomial $c_t(E) = \left(\frac{1}{1-t^2}\right)^k$ which has natural cohomology $H^q(E(l))$ in the range $-3 \leqq l \leqq 0$. These bundles are stable, hence trivial on generic lines and have a symplectic structure which is unique up to multiplication with scalars [14].

Via the Penrose transformation a certain subset of these bundles corresponds to self-dual solutions of the $SU(2)$ Yang-Mills equations on S^4 [1].

Recently the Penrose transformation has been generalized by Salamon [13].

Salamon constructs for every quaternionic manifold M a twistor bundle

$$\pi: Z \to M$$

over M, such that the total space Z is a complex manifold [13]. If $M = \mathbb{P}^n_{\mathbb{H}}$ is the n-dimensional projective space over the quaternions $\mathbb{H}$ this twistor bundle is the well-known fibration

$$\pi: \mathbb{P}^{2n+1}_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{H}}.$$

Identifying $\mathbb{C}^{2n+2}$ with ${}_{\mathbb{C}}\mathbb{H}^{n+1}$ one obtains a real structure δ on $\mathbb{P}^{2n+1}_{\mathbb{C}}$ such that the real lines are precisely the fibres of π.

The special case $M = \mathbb{P}^1_{\mathbb{H}} = S^4$ is just the usual Penrose transformation.

This makes it reasonable to try to construct holomorphic $2n$-bundles on $\mathbb{P}^{2n+1}_{\mathbb{C}}$ using the fibration over $\mathbb{P}^n_{\mathbb{H}}$. Salamon proves the following result [12]:

Let

$$A_i: \mathbb{H}^k \to \mathbb{H}^{n+k}, \quad i = 0, 1, \ldots, n,$$

be $\mathbb{H}$-linear mappings. For every $q = (q_0, \ldots, q_n) \in \mathbb{H}^{n+1}$ define

$$A(q) = \sum_{i=0}^{n} A_i q_i.$$

*) Diese Arbeit entstand während eines Forschungsaufenthalts des zweiten Autors am Max-Planck-Institut für Mathematik in Bonn.

Assume, that for all $q \in \mathbb{H}^{n+1} \setminus \{0\}$ we have

$$(*) \qquad \overline{A(q)}^t A(q) \in GL(k, \mathbb{R}).$$

Then there exists a holomorphic $2n$-bundle E on $\mathbb{P}_{\mathbb{C}}^{2n+1}$ with Chern polynomial $c_t(E) = \left(\frac{1}{1-t^2}\right)^k$, which is trivial on the real lines $\pi^{-1}([q])$ and has a symplectic structure.

Two special cases of this construction are well-known. For $k=1$, $n \geqq 1$ one gets the so called Nullcorrelation bundles [9]. For $n=1$, $k \geqq 1$ this construction gives all mathematical instanton bundles, which come from physics [14].

Unfortunately for $k>1$, $n>1$ the condition $(*)$ is hard to check. Therefore it is not clear, that those bundles E really do exist.

This remark was the starting point for our paper.

Let E be an algebraic rank-$2n$ bundle on $\mathbb{P}_{\mathbb{C}}^{2n+1}$. We say, E is a mathematical instanton bundle with quantum number k, if E is simple, has Chern polynomial $c_t(E) = \left(\frac{1}{1-t^2}\right)^k$ and natural cohomology $H^q(E(l))$ in the range $-2n-1 \leqq l \leqq 0$. Furthermore we require that E has a symplectic structure and trivial splitting type.

We shall prove, that the set $MI_{\mathbb{P}^{2n+1}}(k)$ of isomorphism classes of mathematical instanton bundles with quantum number k on $\mathbb{P}_{\mathbb{C}}^{2n+1}$ can be identified with a quotient

$$MI_{\mathbb{P}^{2n+1}}(k) = SK(2n+2k)/GL(k, \mathbb{C}),$$

where $SK(2n+2k)$ denotes the variety of non-degenerate simple symmetric Kronecker modules of rank $2n+2k$ (Definition 1. 2). From geometric invariant theory it follows that the set $MI^s_{\mathbb{P}^{2n+1}}(k)$ of stable bundles in $MI_{\mathbb{P}^{2n+1}}(k)$ carries the structure of a quasi-projective variety (Theorem 1. 13).

In the second part of the paper we prove that for all $k \geqq 1$, $n \geqq 1$ the moduli spaces $MI_{\mathbb{P}^{2n+1}}(k)$ are non-empty, giving an explicit construction of an appropriate Kronecker module.

1. Properties of mathematical instanton bundles on $\mathbb{P}^{2n+1}$

We use the notation of [9] with some minor changes. Let V be a complex vector space of dimension $2n+2$, $n \geqq 1$, $\mathbb{P} = \mathbb{P}(V^*)$ the associated projective space of lines in V. An algebraic vector bundle E on $\mathbb{P}$ has a *symplectic structure*, if there is an isomorphism

$$\varphi: E \to E^*$$

with $\varphi^* = -\varphi$.

If E is simple, a symplectic structure is unique up to multiplication with scalars.

We say that E has *natural cohomology* in the range $r_1 \leqq l \leqq r_2$, if for every l in that range at most one of the cohomology groups $H^q(E(l))$ is non zero [6].

Definition 1.1. An algebraic rank-$2n$ bundle E on $\mathbb{P}$ is a mathematical instanton bundle with quantum number $k \geqq 1$ if it has the following properties:

(i) the Chern polynomial of E is $c_t(E) = \left(\frac{1}{1-t^2}\right)^k$,

(ii) E has natural cohomology in the range $-2n-1 \leqq l \leqq 0$,

(iii) E has trivial splitting type,

(iv) E is simple,

(v) E has a symplectic structure.

Remark. (i)—(iv) are open properties but not (v) except in the case $n=1$.

We denote the set of isomorphism classes of these bundles by $MI_{\mathbb{P}^{2n+1}}(k)$.

Let H be a complex vector space of dimension k,

$$\alpha : \Lambda^2 V \to L(H, H^*)$$

a linear map. We define the adjoint

$$\hat{\alpha} : V \otimes H \to V^* \otimes H^*$$

of α by

$$\hat{\alpha}(v_1 \otimes h_1)(v_2 \otimes h_2) = \alpha(v_1 \wedge v_2)(h_1)(h_2).$$

For every $v \in V$ let

$$v^{**} : V^* \otimes H^* \to H^*$$

be the evaluation mapping associated to v.

Definition 1.2. A *Kronecker module* on H is a linear map

$$\alpha : \Lambda^2 V \to L(H, H^*)$$

with the following properties

(i) $\hat{\alpha}(v \otimes -) : H \to V^* \otimes H^*$ is injective for all $v \in V \setminus \{0\}$.

(ii) $v^{**} \circ \hat{\alpha} : V \otimes H \to H^*$ is surjective for all $v \in V \setminus \{0\}$.

The *rank* of the Kronecker module α is the rank of the linear map $\hat{\alpha}$.

A Kronecker module α is *symmetric* if the image of α lies in the subspace $S^2 H^* \subset L(H, H^*)$ of the symmetric bilinear forms on H, i.e. if $\hat{\alpha}$ is symplectic.

If for almost all $v_1, v_2 \in V$ the bilinear form $\alpha(v_1 \wedge v_2)$ is non-degenerate we call the Kronecker module α *non-degenerate.*

A Kronecker module α is *simple*, if for each pair $\varphi_1, \varphi_2 \in \operatorname{End} H$ with $\varphi_2^* \alpha = \alpha \varphi_1$ it follows that $\varphi_1 = \varphi_2 = \lambda \operatorname{id}_H$.

A Kronecker module α is called *irreducible* (cf. [7], [12]) if the following condition holds:

If $U \subset H$, $U' \subset H^*$ are linear subspaces, such that $U' \neq 0$, $U' \neq H^*$ and $\alpha(v_1 \wedge v_2)(U) \subset U'$ for linearly independent $v_1, v_2 \in V$, then $\dim U < \dim U'$.

Remark. Property (i) is equivalent to (ii) for symmetric Kronecker modules.

We want to associate to every mathematical instanton bundle with quantum number k on $\mathbb{P}=\mathbb{P}^{2n+1}$ a non-degenerate simple symmetric Kronecker module of rank $2n+2k$.

Lemma 1.3. *Let E be a rank-$2n$ vector bundle with Chern polynomial $c_t(E)=\left(\frac{1}{1-t^2}\right)^k$ on $\mathbb{P}$.*

If E has natural cohomology in the range $-2n-1\leqq l\leqq 0$, E is the cohomology bundle of a monad

(1) $0\to H^{2n}(E(-2n-1))\otimes \mathcal{O}(-1)\to H^1(E(-1))\otimes \Omega(1)\to H^1(E)\otimes\mathcal{O}\to 0.$

Proof. From the Riemann-Roch formula we find the Hilbert polynomial of E

$$\chi(E(l))=2n\binom{l+2n+1}{2n+1}-k\binom{l+2n}{2n-1}.$$

The proof follows now from the Beilinson spectral sequence [9]

$$E_1^{pq}=H^q(E(p))\otimes\Omega^{-p}(-p)\Rightarrow E^{p+q}=\begin{cases}E & \text{for } p+q=0,\\ 0 & \text{for } p+q\neq 0.\end{cases}$$

On $\mathbb{P}=\mathbb{P}(V^*)$ we have the Euler sequence

$$0\to\Omega(1)\to V^*\otimes\mathcal{O}\to\mathcal{O}(1)\to 0.$$

Tensoring this sequence with $H^1(E(-1))$ and combining it with (1) we get the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{2n}(E(-2n-1))\otimes\mathcal{O}(-1) & \xrightarrow{a} & H^1(E(-1))\otimes\Omega(1) & \xrightarrow{b} & H^1(E)\otimes\mathcal{O} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle a'} & & \downarrow & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & H^1(E(-1)\otimes\Omega(1))\otimes\mathcal{O} & \longrightarrow & H^1(E(-1))\otimes V^*\otimes\mathcal{O} & \xrightarrow{u} & H^1(E)\otimes\mathcal{O} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle b'} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(E(-1))\otimes\mathcal{O}(1) & \xrightarrow{=} & H^1(E(-1))\otimes\mathcal{O}(1) & \longrightarrow & 0 & \longrightarrow & 0.\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2}$$

The first row and the first column are monads with the same cohomology E. The remaining rows and columns are exact.

Corollary 1.4. *Let E be a bundle as in* 1.3. *Then E is the cohomology of a monad*

$$0\to H^{2n}(E(-2n-1))\otimes\mathcal{O}(-1)\xrightarrow{a'}H^1(E(-1)\otimes\Omega(1))\otimes\mathcal{O}\xrightarrow{b'}H^1(E(-1))\otimes\mathcal{O}(1)\to 0.$$

Define

$$H(E)=H^{2n}(E(-2n-1)),\quad K(E)=H^1(E(-1)\otimes\Omega(1)).$$

Lemma 1.5. *Let E be a bundle as in 1.3. A symplectic structure $\varphi: E \to E^*$ induces a symplectic structure $q: K(E) \to K(E)^*$ on $K(E)$ such that E is the cohomology bundle of a self-dual monad*

$$(3) \qquad 0 \longrightarrow H(E) \otimes \mathcal{O}(-1) \xrightarrow{a'} K(E) \otimes \mathcal{O} \xrightarrow{a'^* q} H(E)^* \otimes \mathcal{O}(1) \longrightarrow 0.$$

Proof. From [9] it follows that the morphisms of E to E^* correspond to morphisms of the associated monads. So φ induces the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H(E) \otimes \mathcal{O}(-1) \arrow[r, "a'"] \arrow[d, "\varphi_1"] & K(E) \otimes \mathcal{O} \arrow[r, "b'"] \arrow[d, "\varphi_2"] & H^1(E(-1)) \otimes \mathcal{O}(1) \arrow[r] \arrow[d, "\varphi_3"] & 0 \\
0 \arrow[r] & H^1(E(-1))^* \otimes \mathcal{O}(-1) \arrow[r, "b'^*"'] & K(E)^* \otimes \mathcal{O} \arrow[r, "a'^*"'] & H(E)^* \otimes \mathcal{O}(1) \arrow[r] & 0.
\end{tikzcd}$$

Now $q = \varphi_2$ is the induced symplectic structure, and with φ_3 as an identification (Serre duality associated to the given symplectic structure φ) we get $b' = a'^* q$.

Now let E be a rank-$2n$ bundle on $\mathbb{P}$ with the properties (i) (ii) and (v) of definition 1.1. With respect to some symplectic structure $\varphi: E \to E^*$ we get a canonical identification $H^1(E(-1)) \xrightarrow{\sim} H(E)^*$. The morphism a in the monad (1) can then be written as

$$a = a_{E,\varphi}: H(E) \otimes \mathcal{O}(-1) \to H(E)^* \otimes \Omega(1).$$

a is represented by a linear map

$$\hat{\alpha} = \hat{\alpha}_{E,\varphi}: V \otimes H(E) \to V^* \otimes H(E)^*,$$

which is the adjoint of a linear map

$$\alpha = \alpha_{E,\varphi}: \Lambda^2 V \to L(H(E), H(E)^*).$$

Claim. *$\hat{\alpha}$ is symplectic.*

Proof. From (2) and (3) we get the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & H(E) \otimes \mathcal{O}(-1) \arrow[r, "a"] \arrow[d, "a'"] \arrow[dl, "\pi^*"'] & H(E)^* \otimes \Omega(1) \arrow[d] \\
V \otimes H(E) \otimes \mathcal{O} \arrow[r, "\alpha_2"] & K(E) \otimes \mathcal{O} \arrow[r, "\alpha_1"] \arrow[d, "a'^* q"] & V^* \otimes H(E)^* \otimes \mathcal{O} \arrow[d, "\pi"] \\
 & H(E)^* \otimes \mathcal{O}(1) \arrow[r, equal] \arrow[d] & H(E)^* \otimes \mathcal{O}(1) \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

It follows $\pi \alpha_1 = a'^* q = (\alpha_2 \pi^*)^* q = \pi \alpha_2^* q$ and therefore $\alpha_1 = \alpha_2^* q$.

Now by definition $\hat{\alpha}$ is equal to $\alpha_1 \alpha_2$ and thus we have

$$\hat{\alpha}^* = (\alpha_2^* q \alpha_2)^* = \alpha_2^* q^* \alpha_2 = -\hat{\alpha}.$$

So we can consider α as a map $\Lambda^2 V \to S^2 H(E)^*$. We then obtain the following "symplectic" commutative diagram with exact columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H(E)\otimes\mathcal{O}(-1) & \xrightarrow{a} & H(E)^*\otimes\Omega(1) & \xrightarrow{b} & H^1(E)\otimes\mathcal{O} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow H^1(E)^*\otimes\mathcal{O} & \xrightarrow{-\beta^*} & V\otimes H(E)\otimes\mathcal{O} & \xrightarrow{\hat{\alpha}} & V^*\otimes H(E)^*\otimes\mathcal{O} & \xrightarrow{\beta} & H^1(E)\otimes\mathcal{O} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow H^1(E)^*\otimes\mathcal{O} & \xrightarrow[-b^*]{} & H(E)\otimes T(-1) & \xrightarrow[-a^*]{} & H(E)^*\otimes\mathcal{O}(1) & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
0 & & 0 & & 0 & & & &
\end{array} \tag{4}
$$

The second row of this diagram is also exact.

Proposition 1.6. *Let E be a rank-$2n$ bundle on $\mathbb{P}$ with Chern polynomial $c_t(E)=\left(\frac{1}{1-t^2}\right)^k$ and natural cohomology in the range $-2n-1 \leqq l \leqq 0$, $\varphi: E \to E^*$ a symplectic structure on E. Then the associated map*

$$\alpha = \alpha_{E,\varphi}: \Lambda^2 V \to S^2 H(E)^*$$

is a symmetric Kronecker module of rank $2n+2k$. Furthermore we have:

(i) *α is simple if and only if E is simple.*

(ii) *α is non-degenerate if and only if E has trivial splitting type.*

Proof. Since a has to be injective on fibres we see from (4) that α is a symmetric Kronecker module. The rank of $\hat{\alpha}$ is $\dim(V\otimes H(E)) - h^1(E) = 2n+2k$.

(i) follows immediately from

Lemma 1.7. *Let H_i', H_i be complex vector spaces, $i=1, 2, 3$, and*

$$M\ = 0 \to H_1\otimes\mathcal{O}(-1) \xrightarrow{a} H_2\otimes\Omega(1) \xrightarrow{b} H_3\otimes\mathcal{O} \to 0,$$

$$M' = 0 \to H_1'\otimes\mathcal{O}(-1) \xrightarrow{a'} H_2'\otimes\Omega(1) \xrightarrow{b'} H_3'\otimes\mathcal{O} \to 0$$

monads. Let $H^\bullet = \mathrm{Hom}^\bullet(M, M')$ be the following complex. H^i is the complex vector space of all homomorphisms $M \to M'$ of degree i; the differentials $d^i: H^i \to H^{i+1}$ are defined by

$$d^0(x, y, z) = (a'x - ya,\ b'y - zb),$$

$$d^1(x, y) \quad = b'x + ya.$$

Then there exist canonical isomorphisms

$$\mathrm{Ext}^q(E, E') \xrightarrow{\sim} H^q(H^\bullet) \quad for \quad q \geqq 0,$$

where

$$E = \ker b/\mathrm{im}\, a, \quad E' = \ker b'/\mathrm{im}\, a'.$$

Especially we have

$$\mathrm{Hom}(E, E') \cong \ker d^0 = \{\textit{homomorphisms of complexes } M \to M'\}.$$

Proof. [10].

(ii) follows from the following more precise result (cf. [9] II. 4. 2. 3).

Lemma 1.8. *Let E, α be as in proposition 1.6. If $L \subset \mathbb{P}$ is the line defined by $v_1, v_2 \in V$, $v_1 \wedge v_2 \neq 0$, then the restriction E_L of E to L is trivial if and only if the symmetric bilinear form $\alpha(v_1 \wedge v_2)$ on $H(E)$ is non-degenerate, i.e. $\mathrm{rk}\,\alpha(v_1 \wedge v_2) = k$.*

Proof. Let $W \subset V$ be the subspace generated by v_1 and v_2, $\alpha(v_1 \wedge v_2)$ can be considered as linear map

$$\alpha_W = \alpha(v_1 \wedge v_2) : \Lambda^2 W \to S^2 H(E)^*$$

with adjoint

$$(\alpha_W)\hat{}: W \otimes H(E) \to W^* \otimes H(E)^*.$$

Restricting the monad (a, b) in (4) to L and combining with the exact sequence

$$0 \to (V/W)^* \otimes \mathcal{O}_L \to \Omega(1)_L \to \Omega_L(1) \to 0$$

we get the following short exact sequence of complexes of vector bundles on L:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H' \otimes \mathcal{O}_L(-1) \arrow[r, "a'_L"] \arrow[d] & H(E)^* \otimes (V/W)^* \otimes \mathcal{O}_L \arrow[r, "b'_L"] \arrow[d] & H^1(E) \otimes \mathcal{O}_L \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & H(E) \otimes \mathcal{O}_L(-1) \arrow[r, "a_L"] \arrow[d] \arrow[dr, "\tilde{a}_L"] & H(E)^* \otimes \Omega(1)_L \arrow[r, "b_L"] \arrow[d] & H^1(E) \otimes \mathcal{O}_L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H'' \otimes \mathcal{O}_L(-1) \arrow[r] \arrow[d] & H(E)^* \otimes \Omega_L(1) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}.$$

From this we obtain the long exact cohomology sequence

$$0 \to \ker b'_L / \mathrm{im}\, a'_L \to E_L \to \mathrm{coker}\, \tilde{a}_L \to \mathrm{coker}\, b'_L \to 0.$$

One easily sees that E_L is trivial if and only if $\tilde{a}_L$ is surjective. But $\tilde{a}_L$ is nothing else than the map associated to $(a_W)\hat{}$ and so $\tilde{a}_L$ is surjective iff α_W is non-degenerate. This completes the proof of the lemma and of the proposition.

Remark. The above proof also shows that $E_L \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L^{\oplus 2n-2} \oplus \mathcal{O}_L(-1)$ if and only if $\mathrm{rk}\,\alpha(v_1 \wedge v_2) = k-1$.

Now let H, W be fixed complex vector spaces of dimension k, $2n(k-1)$ respectively.

Proposition 1.9. *Let* $\alpha: \Lambda^2 V \to S^2 H^*$ *be a simple symmetric Kronecker module of rank* $2n+2k$ *on* H. *Then* α *defines a monad* $M(\alpha)$

$$(5) \qquad 0 \to H \otimes \mathcal{O}(-1) \xrightarrow{a} H^* \otimes \Omega(1) \xrightarrow{b} W \otimes \mathcal{O} \to 0$$

whose cohomology bundle $E(\alpha)$ *is simple, has Chern polynomial* $c_t(E(\alpha)) = \left(\frac{1}{1-t^2}\right)^k$ *and natural cohomology in the range* $-2n-1 \leqq l \leqq 0$.

Furthermore α *induces a symplectic structure* $\varphi: E(\alpha) \to E(\alpha)^*$ *on* $E(\alpha)$ *such that* $\hat{\alpha} = g^* \hat{\alpha}_{E,\varphi} g$ *with a suitable isomorphism* $g: H \xrightarrow{\sim} H(E)$.

Proof. The first part of the proposition is clear.

From (5) we get a commutative diagram analogous to (4). The corresponding connecting homomorphism

$$\partial: E(\alpha)^* = H^1(M(\alpha)^*) \to H^2(M(\alpha)) = E(\alpha)$$

gives us a symplectic structure $\varphi = \partial^{-1}$. Now the identity $\mathrm{id}: E \to E$ induces isomorphisms $g_1: H \to H(E)$, $g_2: H^* \to H(E)^*$ such that

$$\hat{\alpha}_{E,\varphi} g_1 = g_2 \hat{\alpha}.$$

Since $\hat{\alpha}_{E,\varphi}$ and $\hat{\alpha}$ are symplectic we get $\hat{\alpha}(g_2^* g_1) = g_1^* \hat{\alpha}_{E,\varphi} g_1$ and thus $(g_2^* g_1)^* \hat{\alpha} = \hat{\alpha}(g_2^* g_1)$. By assumption α is simple and so we have $g_2^* g_1 = \lambda^2 \mathrm{id}_H$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Taking $g = \frac{1}{\lambda} g_1$ we are done.

Now let $SK_V(H) \subset L(\Lambda^2 V, S^2 H^*)$ denote the set of all non-degenerate simple symmetric Kronecker modules of rank $2n+2k$. We consider the natural action

$$\alpha^g = g^* \alpha g$$

of $GL(H)$ on $L(\Lambda^2 V, S^2 H^*)$, where $g^* \alpha g$ is defined by

$$g^* \alpha g(v_1 \wedge v_2) = g^* \circ \alpha(v_1 \wedge v_2) \circ g.$$

$SK_V(H)$ is $GL(H)$-invariant.

Proposition 1.10. *The map* $\alpha \mapsto E(\alpha)$ *induces a bijection*

$$\phi: SK_V(H)/GL(H) \to MI_{\mathbb{P}}(k).$$

Proof. If E is a mathematical instanton bundle, $\varphi: E \to E^*$ a symplectic structure on E, $g: H \to H(E)$ an isomorphism, then $\hat{\alpha} = g^* \hat{\alpha}_{E,\varphi} g$ defines a Kronecker module $\alpha \in SK_V(H)$ with $E(\alpha) \cong E$, thus ϕ is surjective. The injectivity of ϕ follows by the same argument as at the end of the proof of proposition 1.9.

Now we want to show, that the set $MI^s_{\mathbb{P}^{2n+1}}(k)$ of isomorphism classes of *stable* mathematical instanton bundles with quantum number k carries the structure of a quasi-projective variety.

Let $P = \mathbb{P}(L(\Lambda^2 V, S^2 H^*)^*)$ be the projective space of lines in $L(\Lambda^2 V, S^2 H^*)$.

We consider the closed subspace

$$X \subset P$$

consisting of all points $[\alpha] \in P$ which satisfy the rank condition

$$\operatorname{rk} \hat{\alpha} \leqq 2n + 2k.$$

X is $SL(H)$-invariant under the natural action of $SL(H)$ on P. Let $X^s(X^{ss})$ be the open set of (semi-)stable points in X with respect to $SL(H)$ in the sense of Mumford [8]. Then the quotient $X^{ss}/SL(H)$ exists and is a projective variety. $X^s/SL(H)$ is an open subspace of $X^{ss}/SL(H)$ [8]. In order to show that $MI^s_{\mathbb{P}^{2n+1}}(k)$ is an open subset of $X^s/SL(H)$ we need the following two lemmata.

Lemma 1.11. *Let $\alpha \in SK_V(H)$ be a Kronecker module, $E = E(\alpha)$ the associated instanton bundle. If E is stable, then α is irreducible.*

Proof. First we recall that E is stable if there doesn't exist any subsheaf $F \subset E$ with $0 < \operatorname{rk} F < \operatorname{rk} E$ and $c_1(F) \geqq 0$. The proof is now essentially the same as the proof of Le Potier [12] and so we omit it.

Lemma 1.12. *A Kronecker module $\alpha \in SK_V(H)$ is irreducible if and only if the point $[\alpha] \in X$ is stable with respect to $SL(H)$.*

Proof. Again we omit the proof since the proof of Hulek [7] generalizes without difficulty to our case.

Now let $SK^s_V(H) \subset SK_V(H)$ be the set of Kronecker modules belonging to stable bundles, $\mathbb{P}(SK^s_V(H))$ the corresponding $SL(H)$-invariant open subset of X.

From Lemma 1.11 and Lemma 1.12 we know that $\mathbb{P}(SK^s_V(H)) \subset X^s$ and we get

Theorem 1.13. *The map $\alpha \mapsto E(\alpha)$ induces a bijection*

$$\psi : \mathbb{P}(SK^s_V(H))/SL(H) \to MI^s_{\mathbb{P}^{2n+1}}(k).$$

ψ induces the structure of a quasi-projective variety on $MI^s_{\mathbb{P}^{2n+1}}(k)$. With this structure $MI^s_{\mathbb{P}^{2n+1}}(k)$ is a coarse moduli space for stable mathematical instanton bundles with quantum number k on $\mathbb{P}^{2n+1}$. $\overline{MI}^s_{\mathbb{P}^{2n+1}}(k) = X^{ss}/SL(H)$ is a natural compactification of $MI^s_{\mathbb{P}^{2n+1}}(k)$.

Let $\mathbb{G} = \operatorname{Grass}_2(V)$ be the grassmannian of lines in $\mathbb{P}$,

$$\mathbb{G} = \{[v_1 \wedge v_2] \mid v_1, v_2 \in V, v_1 \wedge v_2 \neq 0\} \subset \mathbb{P}(\Lambda^2 V^*).$$

Let α be an element of $SK_V(H)$.

We have the canonical inclusion

$$H \otimes \mathcal{O}_{\mathbb{G}}(-1) \to H \otimes \Lambda^2 V \otimes \mathcal{O}_{\mathbb{G}}.$$

The composition with

$$\alpha : H \otimes \Lambda^2 V \otimes \mathcal{O}_{\mathbb{G}} \to H^* \otimes \mathcal{O}_{\mathbb{G}}$$

defines a morphism

$$\theta_\alpha : H \otimes \mathcal{O}_{\mathbb{G}}(-1) \to H^* \otimes \mathcal{O}_{\mathbb{G}}.$$

Since α is non-degenerate θ_α is a monomorphism and

$$\Theta(\alpha) = \operatorname{coker} \theta_\alpha(-1)$$

is a sheaf on $\mathbb{G}$ with support on the set $S_{E(\alpha)}$ of jumping lines of $E(\alpha)$.

We call $\Theta(\alpha)$ the *theta-characteristic* associated to α.

Lemma 1. 14. *Let* $\alpha, \alpha' \in SK_V(H)$ *be Kronecker modules with associated theta-characteristics* Θ, Θ'.

Θ *and* Θ' *are isomorphic if and only if* α *and* α' *lie in the same* $GL(H)$*-orbit.*

Proof. From $\alpha' = g^* \alpha g$ we get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H \otimes \mathcal{O}_{\mathbb{G}}(1) \arrow[r] \arrow[d, "g^{-1}"] & H^* \otimes \mathcal{O}_{\mathbb{G}} \arrow[r] \arrow[d, "g^*"] & \Theta(1) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H \otimes \mathcal{O}_{\mathbb{G}}(-1) \arrow[r] & H^* \otimes \mathcal{O}_{\mathbb{G}} \arrow[r] & \Theta^1(1) \arrow[r] & 0.
\end{tikzcd}$$

Conversely an isomorphism $\psi: \Theta \to \Theta'$ induces isomorphisms $g_2: H \to H$, $g_1: H^* \to H^*$ such that $\theta_{\alpha'} g_2 = g_1 \theta_\alpha$ and thus $\hat{\alpha}' g_2 = g_1 \hat{\alpha}$. Since α' is simple we get $\alpha' = g^* \alpha g$ if we put $g = \frac{1}{\lambda} g_1^*$ with a suitable scalar λ.

Remark. $S_{E(\alpha)} \subset \mathbb{G}$ is a hypersurface of degree k with equation $\det \alpha(v_1 \wedge v_2) = 0$.

Since $E(\alpha)$ always has jumping lines of higher order [4] the sheaf $\Theta(\alpha)$ can't be invertible on $S_{E(\alpha)}$. But one can easily show that $\Theta(\alpha)$ has the following properties which justify our terminology:

i) $\Theta(\alpha)$ is an $\mathcal{O}_{S_{E(\alpha)}}$-sheaf.

ii) $\Theta(\alpha) \cong \underline{\operatorname{Ext}}^1_{\mathcal{O}_{\mathbb{G}}}(\Theta(\alpha), \mathcal{O}_{\mathbb{G}}(-3))$.

iii) The restriction of $\Theta(\alpha)$ to the dual plane $\mathbb{P}^* \subset \mathbb{G}$ of a generic plane $\mathbb{P} \subset \mathbb{P}^{2n+1}$ is a theta characteristic on the curve of jumping lines of $E|\mathbb{P}$ in the sense of Barth [3].

From proposition 1. 10 we see that we can define a theta-characteristic θ_E for every mathematical instanton bundle E. θ_E determines E up to isomorphism.

2. Existence of mathematical instanton bundles on $\mathbb{P}^{2n+1}$

The purpose of this section is to show that the sets $MI_{\mathbb{P}^{2n+1}}(k)$ are non-empty for all $k \geqq 1$, $n \geqq 1$.

Proposition 1. 10 shows, that it is sufficient to construct a non-degenerate simple symmetric Kronecker module α of rank $2n+2k$. By definition α is a linear map

$$\alpha: \Lambda^2 V \to S^2 H^*.$$

We choose a basis in H and represent α by a $k \times k$-matrix A with entries in $\Lambda^2 V^*$,

$$A = (A_{ij})_{i,j=1,\ldots,k}, \quad A_{ij} \in \Lambda^2 V^*.$$

First we have to express the properties of α in terms of A. Identifying $\Lambda^2 V^*$ with the space of symplectic linear maps

$$\Lambda^2 V^* = \{\varphi \in L(V, V^*) \mid \varphi^* = -\varphi\}$$

we define for every $v \in V^*$ a vector

$$A_i(v) = \begin{pmatrix} A_{i1}(v) \\ \vdots \\ A_{ik}(v) \end{pmatrix} \in V^{*\oplus k}.$$

We then get

Lemma 2. 1. *Let $\alpha : \Lambda^2 V \to L(H, H^*)$ be a linear map, $A = (A_{ij})$ a matrix, which represents α with respect to a basis of H. α is a symmetric Kronecker module of rank $2n+2k$ if and only if A has the following properties:*

(i) $A_{ij} = A_{ji} \ \forall i, j$.

(ii) *For all $v \in V \setminus \{0\}$ we have in $\Lambda^k(V^{*\oplus k})$*

$$A_1(v) \wedge \cdots \wedge A_k(v) \neq 0.$$

(iii) $\operatorname{rk} A = 2n + 2k$, *where we consider A as a linear map*

$$A : V^{\oplus k} \text{---} V^{*\oplus k}.$$

α is non-degenerate iff the following holds:

(iv) $\operatorname{rk}(A_{ij}(v_1 \wedge v_2)) = k$ *for almost all $v_1, v_2 \in V$.*

α is simple iff A has the property

(v) $AX = YA$ *for complex $k \times k$-matrices X, Y implies $X = Y = \lambda I_k$.*

Now let A be a matrix with the properties (i)—(iii). Then A defines a monad

(6) $$0 \to \mathcal{O}(-1)^{\oplus k} \xrightarrow{a} \Omega(1)^{\oplus k} \xrightarrow{b} \mathcal{O}^{\oplus m} \to 0$$

where $M = 2n(k-1)$.

The morphism a is given by A and b is given by an $m \times k$-matrix

$$B = (v_{ij}) \quad \begin{matrix} i = 1, \ldots, m, \\ j = 1, \ldots, k \end{matrix} \quad v_{ij} \in V,$$

with entries in V.

Lemma 2. 2. *A matrix $B = (v_{ij})$ defines an epimorphism $b : \Omega(1)^{\oplus k} \to \mathcal{O}^{\oplus m}$ if and only if for all $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m \setminus \{0\}$ we have*

(vi) $$\left(\sum_{\mu=1}^{m} \lambda_\mu v_{\mu i}\right) \wedge \left(\sum_{\mu=1}^{m} \lambda_\mu v_{\mu j}\right) \neq 0$$

for at least on pair $1 \leqq i, j \leqq k$.

Proof. b is an epimorphism if and only if b^* is injective in each fibre. This is condition (vi).

Proposition 2.3. *Let $k \geqq 2$. Choose a basis $\{e_1, \ldots, e_{n+1}, f_1, \ldots, f_{n+1}\}$ for V. We define*

$$e = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}, \quad e' = \begin{pmatrix} e_2 \\ \vdots \\ e_{n+1} \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \quad f' = \begin{pmatrix} f_2 \\ \vdots \\ f_{n+1} \end{pmatrix}$$

and

$$(7) \qquad B = \begin{pmatrix} e & e' & & & & \\ f & f' & & & & \\ & e & e' & & & \\ & f & f' & & & \\ & & & \ddots & & \\ & & & & e & e' \\ & & & & f & f' \end{pmatrix}.$$

Then B defines an epimorphism $b : \Omega(1)^{\oplus k} \to \mathcal{O}^{\oplus m}$.

Proof. We have to verify the condition (vi).

Let

$$\lambda_1 = (\lambda_1^i, \ldots, \lambda_n^i) \in \mathbb{C}^n, \ \mu_i = (\mu_1^i, \ldots, \mu_n^i) \in \mathbb{C}^n, \ i = 1, \ldots, k-1,$$
$$x = (\lambda_1, \mu_1, \ldots, \lambda_{k-1}, \mu_{k-1}) \in \mathbb{C}^m.$$

If B_i denotes the i^{th} column of B we must show that

$$xB_1 \wedge xB_j = 0 \in \Lambda^2 V \quad \forall\, i < j$$

implies $x = 0$.

Define $\lambda_j^i = 0$, $\mu_j^i = 0$ if $j \leqq 0$ or $j \geqq n+1$ or $i \leqq 0$ or $i \geqq k$. Then we compute

$$xB_i = \sum_{\nu=1}^{n+1} (\lambda_{\nu-1}^{i-1} + \lambda_\nu^i)\, e_\nu + \sum_{\nu=1}^{n+1} (\mu_{\nu-1}^{i-1} + \mu_\nu^i)\, f_\nu .$$

Assume now $xB_i \wedge xB_j = 0 \ \forall\, i, j$. We show by induction, that then $\lambda_i = 0$.

This is true for $i \leqq 0$ by definition. For the induction step we assume $\lambda_\nu^{i-1} = 0$ for all ν and show $\lambda_\nu^i = 0$ using descending induction on ν. If $\lambda_{\nu+j}^i$ vanishes for $j \geqq 1$ the coefficient of $e_\nu \wedge e_{\nu+j}$ in $xB_i \wedge xB_{i+j}$ $(j = 1, \ldots, n+1-\nu)$ is

$$\lambda_\nu^i (\lambda_{\nu+j-1}^{i+j-1} + \lambda_{\nu+j}^{i+j}).$$

These coefficients vanish. We form the alternating sum and get

$$(\lambda_\nu^i)^2 = \sum_{j \geqq 1} (-1)^{j+1} \lambda_\nu^i (\lambda_{\nu+j-1}^{i+j-1} + \lambda_{\nu+j}^{i+j}) = 0.$$

This proves the proposition.

If we can find a matrix $A = (A_{ij}) \in (\Lambda^2 V^*)^{k \times k}$ which has the three properties

(i) $A_{ij} = A_{ji}$,

(ii) $A_1(v) \wedge \cdots \wedge A_k(v) \neq 0$ for $v \in V \setminus \{0\}$,

(iii)′ $BA = 0$,

then A will define a symmetric Kronecker module of rank $2n+2k$.

Consider the vector space P_B of matrices $A \in (\Lambda^2 V^*)^{k \times k}$ with (i) and (iii)′. It is easy to define a basis for this vector space.

Proposition 2. 4. *Let* $z = (z_1, \ldots, z_{2n+2k-1}) \in \mathbb{C}^{2n+2k-1}$ *and*

$$A'_j(z) = \begin{pmatrix} z_{2k-j} & z_{2k-j+1} & \cdots & z_{2k-j+n} \\ z_{2k-j+1} & & . \cdot & \\ \vdots & . \cdot & & \vdots \\ z_{2k-j+n} & \cdots & & z_{2k-j+2n} \end{pmatrix},$$

$$A_j(z) = (-1)^j \begin{pmatrix} 0 & -A'_j(z) \\ A'_j(z) & 0 \end{pmatrix},$$

$$A(z) = \begin{pmatrix} A_1(z) & A_2(z) & \cdots & A_k(z) \\ A_2(z) & & . \cdot & \\ \vdots & . \cdot & & \vdots \\ A_k(z) & & \cdots & A_{2k-1}(z) \end{pmatrix}.$$

The map $z \mapsto A(z)$ *is an isomorphism* $\mathbb{C}^{2n+2k-1} \to P_B$.

Proof. Identifying A_{ij} with a skew-symmetric $(2n+2) \times (2n+2)$-matrix condition (iii)′ means: the ν^{th} column of A_{ij} equals the $(\nu+1)^{\text{th}}$ column of $-A_{i+1,j}$ for $\nu = 1, \ldots, n$ and $\nu = n+2, \ldots, 2n+1$.

The proof is now straightforward.

Now we use this isomorphism to define A. If $\{\varepsilon_1, \ldots, \varepsilon_{2n+2k-1}\}$ is the standard basis of $\mathbb{C}^{2n+2k-1}$ we define

$$A = \begin{cases} A(\varepsilon_{2k-1} + \varepsilon_{3k+n-1}) & \text{for } k \leqq n, \\ A(\varepsilon_{2k-1}) & \text{for } k = n+1, \\ A(\varepsilon_{k-n-1} + \varepsilon_{2k-1}) & \text{for } k > n+1. \end{cases} \tag{8}$$

Proposition 2. 5. *The matrix A defined in* (8) *has the property* (ii).

Proof. It is sufficient to prove that the equation

$$\begin{pmatrix} A'_1\lambda \\ \vdots \\ A'_k\lambda \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} A'_k\lambda \\ \vdots \\ A'_{2k-1}\lambda \end{pmatrix} = 0, \ \lambda \in \mathbb{C}^{n+1}$$

has only the trivial solution $\lambda = 0$. This is equivalent to the following

Claim. *If all k-minors of the $k(n+1) \times k$-matrix*

$$A'\lambda = \begin{pmatrix} A'_1\lambda & A'_2\lambda & \cdots & A'_k\lambda \\ A'_2\lambda & & & \\ \vdots & & & \vdots \\ A'_k\lambda & & \cdots & A'_{2k-1}\lambda \end{pmatrix}$$

vanish, it follows, that $\lambda = 0$.

To prove this claim, one has to consider the two cases $k \leqq n+1$, $k > n+1$ separately. Writing out the matrices $A'\lambda$ in each of these two cases it is only a matter of patience to check the claim.

We can now use the matrix A in (8) to construct an algebraic rank-$2n$ bundle E_A on $\mathbb{P}^{2n+1}$ with Chern polynomial $c_t(E_A) = \left(\frac{1}{1-t^2}\right)^k$. E_A has a symplectic structure and natural cohomology in the range $-2n-1 \leqq l \leqq 0$. It remains to verify, that E_A is simple and trivial on generic lines, i.e. that A has the properties (v) and (iv) in lemma 2. 1. (v) can be checked directly. To prove (iv), it is sufficient to find some special vectors $v_1 = \sum a_i e_i$, $v_2 = \sum b_i f_i$ such that the $k \times k$-matrix

$$A(v_1 \wedge v_2) = \begin{pmatrix} -b^t A'_1 a & b^t A'_2 a & \cdots & \pm b^t A'_k a \\ b^t A'_2 a & & \iddots & \\ \vdots & & & \vdots \\ \pm b^t A'_k a & & \cdots & -b^t A'_{2k-1} a \end{pmatrix}$$

is non-degenerate.

For example if $k \leqq n+1$ we get

$$A(e_1 \wedge f_k) = \pm \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}.$$

The case $k > n+1$ is similar.

This was the final step in proving

Theorem 2. 6. *For every $k \geqq 1$, $n \geqq 1$ there exist mathematical instanton bundles with quantum number k on $\mathbb{P}^{2n+1}$.*

The above construction gives only very special bundles E on $\mathbb{P}^{2n+1}$. For these bundles we have $h^0(E(1)) = 2n$ and the evaluation map $\varphi : \mathcal{O}^{\oplus 2n} \to E(1)$ has the determinant

$$\det \varphi = \det \begin{pmatrix} x_1 & \cdots & x_{n+1} & & \\ & \ddots & & & \\ & & x_1 & \cdots & x_{n+1} \\ y_1 & \cdots & y_{n+1} & & \\ & \ddots & & & \\ & & y_1 & \cdots & y_{n+1} \end{pmatrix} \in H^0(\mathcal{O}(2n))$$

where $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}$ are homogeneous coordinates on $\mathbb{P}^{2n+1}$ as above.

$X = (\det \varphi)_0$ is an irreducible hypersurface of degree $2n$ in $\mathbb{P}^{2n+1}$ and non singular only in the case $n = 1$. $E(1)$ can be constructed as extension

$$0 \to \mathcal{O}^{2n-1} \to E(1) \to J_Y(2n) \to 0$$

with a 2-codimensional subspace $Y \subset \mathbb{P}^{2n+1}$ lying in X.

For example in the case $n = 2$, $k = 2$, Y is a variety of degree 8 in $\mathbb{P}^5$.

It would be interesting to compute the moduli space $MI_{\mathbb{P}^5}(2)$. In particular one should try to decide if there are smooth varieties Y which occur as dependency locus of sections in $E(1)$. The generic hyperplane section S of Y would be a smooth rational surface of degree 8 in $\mathbb{P}^4$ of type $S = \tilde{\mathbb{P}}^2(x_0, \ldots, x_{10})$ embedded by the linear system of degree 7 curves in $\mathbb{P}^2$ with nodes in $x_1, \ldots, x_{10}$ passing through x_0 [11].

References

[1] *M. F. Atiyah, R. S. Ward,* Instantons and algebraic geometry, Comm. Math. Phys. **55** (1977), 117—124.

[2] *W. Barth,* Irreducibility of the space of mathematical instanton bundles with rank 2 and $c_2 = 4$, Math. Ann. **258** (1981), 81—106.

[3] *W. Barth,* Moduli of vector bundles on the projective plane, Invent. math. **42** (1977), 63—91.

[4] *J. Brun, A. Hirschowitz,* Variété des droites sauteuses du fibré instanton général, Compos. Math. **53** (1984), 325—336.

[5] *R. Hartshorne,* Stable vector bundles and instantons, Comm. Math. Phys. **59** (1978), 1—15.

[6] *R. Hartshorne, A. Hirschowitz,* Cohomology of a general instanton bundle, Ann. scient. Ec. Norm. Sup. (4) **15** (1982), 365—390.

[7] *K. Hulek*, On the classification of stable rank-r vector bundles over the projective plane. Vector bundles and differential equations, Proceedings Nice 1979, Progress in Math. **7**, Boston 1980.
[8] *D. Mumford*, Geometric invariant theory, Berlin-Heidelberg-New York 1965.
[9] *C. Okonek, M. Schneider, H. Spindler*, Vector bundles on complex projective spaces, Progress in Math. **3**, Boston 1980.
[10] *C. Okonek, H. Spindler*, Stabile reflexive Garben vom Rang 3 auf $\mathbb{P}^3$ mit kleinen Chernklassen, Math. Ann. **264** (1983), 91—118.
[11] *C. Okonek*, Flächen vom Grad 8 im $\mathbb{P}^4$, to appear in Math. Z.
[12] *Le Potier*, Sur l'espace de modules des fibres de Yang et Mills, Sém. E.N.S. 1979—1982, Progress in Math. **37**, Boston 1983.
[13] *S. M. Salamon*, Quaternionic structures and twistor spaces, in: Global Riemannian geometry, Chichester 1984.
[14] *J. L. Verdier*, Instantons, in: Les équations de Yang et Mills, Sém. E.N.S. 1977—1978, Astérisque **71—72** (1980).

Mathematisches Institut der Universität, Bunsenstr. 3—5, D-3400 Göttingen

Eingegangen 1. April 1985